TRANSFER FUNCTIONS FOR THE TEMPERATURE OF CURVILINEAR WALLS AND HOLLOW BODIES UNDER GENERALIZED THERMAL EFFECTS

L. B. Andreeva and N. A. Yaryshev

UDC 536.2.01

We obtain in operator form the solution of a heat-transfer problem for bodies under generalized thermal effects and determine the structure of the transfer functions. We propose approximate equations for the interaction between the effects in order to calculate the mean volume and mean surface temperatures of the body.

We consider the problem of thermal conductivity for a curvilinear wall or a hollow body formed by two convex polygons. The walls (body) are made of a homogeneous isotropic material, have volume V, internal surface S_1, external surface S_2, and are subject to the influence of the following state factors:

1) liquid (gaseous) media with temperatures $t_1(\tau)$ and $t_2(\tau)$ in contact, respectively, with the surfaces S_1 and S_2 of the body;

2) external (superficial) energy sources $q_1(\tau)$ and $q_2(\tau)$ on the surfaces S_1 and S_2;

3) an internal volume energy source $w(\tau)$ in the volume V;

4) internal ("convection") energy sinks, the specific intensity of which, b, is directly proportional to the difference between the local temperature u at the given point of the body and the temperature $w(\tau)$ of the medium permeating the body.

It is assumed that the thermophysical properties and heat-transfer coefficients of the body are constant and the energy sources are distributed uniformly over its surface and throughout its volume.

We know [1] that the analytic solution of three-dimensional problems in heat conduction are quite complex for the above thermal effects. The form of the solution depends on the shape of the body and the nature of the change in the effects with time. The mathematical formulation of the problem can be significantly simplified if, in deriving the heat conduction equation, we consider the temperature u of the body as a function of one generalized coordinate r, uniquely related to the surface of average temperature δ inside the body. In this approach the heat transfer inside the body is determined by the equation of heat conduction

$$\frac{\partial u}{\partial \tau} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{\sigma}\cdot\frac{d\sigma}{dr}\cdot\frac{\partial u}{\partial r}\right) - \frac{b}{c\gamma}(u - v) + \frac{w}{c\gamma}. \tag{1}$$

The equation for the interaction between the surface of average temperature and the generalized coordinate should be specified in the form

$$\sigma(r) = Ar^n. \tag{2}$$

The choice of (2) is determined by the fact that Eq. (1) can be reduced to a form convenient for subsequent integration, since

$$\frac{1}{\sigma}\cdot\frac{d\sigma}{dr} = \frac{n}{r}. \tag{3}$$

Institute of Precision Mechanics and Optics, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 19, No. 6, pp. 1110-1117, December, 1970. Original article submitted September 16, 1969.

In addition, for n = 0, 1, and 2, Eq. (1) becomes the exact equation for one-dimensional problems in heat conduction [1] for bodies of canonical form (flat plates, cylinders, spheres).

The integral properties of the body (external surfaces and volume) are determined, noting (2), from the equations

$$S_1 = Ar_1^n,\ S_2 = Ar_2^n,\ V = \frac{A}{n+1}\left(r_2^{n+1} - r_1^{n+1}\right). \tag{4}$$

We do not discuss here the problems of determining the characteristic dimensions r_1 and r_2 of the body or the form vector. Various approaches to their choice are described in [2].

Under the above assumptions, boundary conditions of the following form hold for Eq. (1), taking account of (3):

$$\begin{gathered} -\lambda \left.\frac{\partial u}{\partial r}\right|_{r=r_2} + q_2 = \alpha_2 \left(u|_{r=r_2} - t_2\right), \\ \lambda \left.\frac{\partial u}{\partial r}\right|_{r=r_1} + q_1 = \alpha_1 \left(u|_{r=r_1} - t_1\right). \end{gathered} \tag{5}$$

At the initial moment of time the temperature distribution inside the body is assumed to be uniform

$$u(r,\ \tau)|_{\tau=0} = 0. \tag{6}$$

The solution of Eq. (1), for the Laplace transformed temperature U(r, s) of the body under conditions (2), (5), and (6), can be expressed in terms of transfer functions and generalized thermal effects in the form

$$U(r,\ s) = Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3, \tag{7}$$

where

$$Z_1 = T_1 + \frac{1}{\alpha_1} Q_1,\ Z_2 = T_2 + \frac{1}{\alpha_2} Q_2, \tag{8}$$

$$Z_3 = V + \frac{1}{b} W. \tag{9}$$

The transfer functions have the following structure:

$$Y_1 = -\frac{\left(\frac{\beta}{\beta_1}\right)^{-\nu} \left[G_2 I_\nu(\beta) - G_3 K_\nu(\beta)\right]}{\Phi}, \tag{10}$$

$$Y_2 = \frac{\left(\frac{\beta}{\beta_2}\right)^{-\nu} \left[G_1 I_\nu(\beta) - G_4 K_\nu(\beta)\right]}{\Phi}, \tag{11}$$

$$Y_3 = \frac{b}{\lambda\mu^2}\left(1 - Y_1 - Y_2\right). \tag{12}$$

Here

$$\beta = \mu r,\ \beta_1 = \mu r_1,\ \beta_2 = \mu r_2,$$

$$\mu = \sqrt{\frac{s}{a} + \frac{\delta}{\lambda}},\ \nu = \frac{n-1}{2},$$

$$\Phi = f_1 + \frac{\beta_2}{\xi_2} f_2 + \frac{\beta_1}{\xi_1} f_3 + \frac{\beta_1\beta_2}{\xi_1\xi_2} f_4,$$

$$G_1 = K_\nu(\beta_1) + \frac{\beta_1}{\xi_1} K_{\nu+1}(\beta_1),$$

$$G_2 = K_\nu(\beta_2) - \frac{\beta_2}{\xi_2} K_{\nu+1}(\beta_2),$$

$$G_3=I_\nu(\beta_2)+\frac{\beta_2}{\xi_2}I_{\nu+1}(\beta_2),$$

$$G_4=I_\nu(\beta_1)-\frac{\beta_1}{\xi_1}I_{\nu+1}(\beta_1),$$

$$f_1=I_\nu(\beta_2)K_\nu(\beta_1)-I_\nu(\beta_1)K_\nu(\beta_2),$$

$$f_2=I_{\nu+1}(\beta_2)K_\nu(\beta_1)+I_\nu(\beta_1)K_{\nu+1}(\beta_2),$$

$$f_3=I_\nu(\beta_2)K_{\nu+1}(\beta_1)+I_{\nu+1}(\beta_1)K_\nu(\beta_2),$$

$$f_4=I_{\nu+1}(\beta_2)K_{\nu+1}(\beta_1)-I_{\nu+1}(\beta_1)K_{\nu+1}(\beta_2), \tag{13}$$

$$\xi_1=\frac{\alpha_1 r_1}{\lambda},\quad \xi_2=\frac{\alpha_2 r_2}{\lambda};$$

$$I_\nu(z)=\sum_{m=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)},$$

$$K_\nu(z)=\frac{\pi}{2\sin(\nu\pi)}\left[I_{-\nu}(z)-I_\nu(z)\right]. \tag{14}$$

To make the transition from (7) to the true temperature of the body, $u(r, \tau)$ is complex and laborious in the general case. To obtain approximate equations linking the typical temperatures of the body and the influences it is permissible to replace the exact transfer functions (10)-(12) by approximate expressions based on the expansions (14). Methods of representing the transforms of the transfer functions in rational fractional form were given in [1-4].

Because the series converge rapidly the replacement of the exact expression for a transfer function by an approximate one produces the most tangible results when the temperatures are calculated at the initial moments of time and the effect of the initial conditions is felt. As time passes the replacement becomes barely noticeable and the approximate expression more accurately reflects the particular features of the original transfer function as the number of terms retained in the series increases.

Let us consider an important special case in heat transfer when there are no internal energy sinks in the body, i.e., $b = 0$. As follows from (13), the parameter $\mu = \sqrt{s/a}$, while the effect Z_3 and the transfer function Y_3 can be rewritten as

$$Z_3=\frac{r_2^2}{\lambda}W,\quad Y_3=\frac{1}{\beta_2^2}(1-Y_1-Y_2), \tag{15}$$

the form of the remaining expressions remaining unaltered.

In practical calculations the transfer functions (10), (11), and (15) can be replaced by approximate expressions of the following form:

$$Y_1=\frac{B_0+B_1 s}{1+A_1 s+A_2 s^2},\quad Y_2=\frac{C_0+C_1 s}{1+A_1 s+A_2 s^2},$$

$$Y_3=\frac{a}{r_2^2}\left(\frac{D_1+D_2 s}{1+A_1 s+A_2 s^2}\right), \tag{16}$$

the coefficients of which are found from the equations

$$A_1=\frac{1}{E}\cdot\frac{r_2^2}{4a}\left\{\left[\frac{1}{1-\nu}+\frac{2}{\xi_2}+\rho_1^2\frac{1}{1+\nu}\right.\right.$$

$$\left.\left.\times\left(1-\frac{2}{\xi_1}\right)\left(1-\frac{2\nu}{\xi_2}\right)\right]-\rho_1^{-2\nu}\left[\frac{1}{1+\nu}\left(1+\frac{2}{\xi_2}\right)\left(1+\frac{2\nu}{\xi_1}\right)+\rho_1^2\frac{1}{1-\nu}\left(1-\frac{2(1-\nu)}{\xi_1}\right)\right]\right\}, \tag{17}$$

$$E=1-\frac{2\nu}{\xi_2}-\rho_1^{-2\nu}\left(1+\frac{2\nu}{\xi_1}\right), \tag{18}$$

$$A_2=\frac{1}{E}\cdot\frac{r_2^4}{16a^2}\left\{\left[-\frac{1}{2(2-\nu)(1-\nu)}\left(1+\frac{2(2-\nu)}{\xi_2}\right)+\rho_1^2\frac{1}{1-\nu^2}\left(1+\frac{2(1-\nu)}{\xi_2}\right)\left(1-\frac{2}{\xi_1}\right)\right.\right. \tag{19}$$

$$+\rho_1^4\frac{1}{2(2+\nu)(1+\nu)}\left(1-\frac{2\nu}{\xi_2}\right)\left(1-\frac{4}{\xi_1}\right)\Bigg]-\rho_1^{-2\nu}\left[\frac{1}{2(2+\nu)(1+\nu)}\left(1+\frac{2\nu}{\xi_1}\right)\left(1+\frac{4}{\xi_2}\right)\right. \tag{20}$$

$$\left.\left.+\rho_1^2\frac{1}{1-\nu^2}\left(1-\frac{2(1-\nu)}{\xi_1}\right)\left(1+\frac{2}{\xi_2}\right)+\rho_1^4\frac{1}{2(2-\nu)(1-\nu)}\left(1-\frac{2(2-\nu)}{\xi_1}\right)\right]\right\}, \tag{21}$$

$$B_0=\frac{1}{E}\left(1-\frac{2\nu}{\xi_2}-\rho^{-2\nu}\right), \tag{22}$$

$$B_1=\frac{1}{E}\cdot\frac{r_2^2}{4a}\left\{\left[\frac{1}{1-\nu}+\frac{2}{\xi_2}+\rho^2\frac{1}{1+\nu}\left(1-\frac{2\nu}{\xi_2}\right)\right]-\rho^{-2\nu}\left[\frac{1}{1+\nu}\left(1+\frac{2}{\xi_2}\right)+\rho^2\frac{1}{1-\nu}\right]\right\}, \tag{23}$$

$$B_2=\frac{1}{E}\cdot\frac{r_2^4}{16a^2}\left\{\left[\frac{1}{2(2-\nu)(1-\nu)}\left(1+\frac{2(2-\nu)}{\xi_2}\right)+\rho^2\frac{1}{1-\nu^2}\left(1+\frac{2(1-\nu)}{\xi_2}\right)\right.\right.$$

$$\left.+\rho^4\frac{1}{2(2+\nu)(1+\nu)}\left(1-\frac{2\nu}{\xi_2}\right)\right]-\rho^{-2\nu}\left[\frac{1}{2(2+\nu)(1+\nu)}\left(1+\frac{4}{\xi_2}\right)\right.$$

$$\left.\left.+\rho^2\frac{1}{1-\nu^2}\left(1+\frac{2}{\xi_2}\right)+\rho^4\frac{1}{2(2-\nu)(1-\nu)}\right]\right\}, \tag{24}$$

$$C_0=\frac{1}{E}\left[-\rho_1^{-2\nu}\left(1+\frac{2\nu}{\xi_1}\right)+\rho^{-2\nu}\right], \tag{25}$$

$$C_1=\frac{1}{E}\cdot\frac{r_2^2}{4a}\left\{-\rho_1^{-2\nu}\left[\rho_1^2\frac{1}{1-\nu}\left(1-\frac{2(1-\nu)}{\xi_1}\right)+\rho^2\frac{1}{1+\nu}\left(1+\frac{2\nu}{\xi_1}\right)\right]\right.$$

$$\left.+\rho^{-2\nu}\left[\rho_1^2\frac{1}{1+\nu}\left(1-\frac{2}{\xi_1}\right)+\rho^2\frac{1}{1-\nu}\right]\right\}, \tag{26}$$

$$C_2=\frac{1}{E}\cdot\frac{r_2^4}{16a^2}\left\{\rho^{-2\nu}\left[\rho_1^2\rho^2\frac{1}{1-\nu^2}\left(1-\frac{2}{\xi_1}\right)\right.\right.$$

$$\left.+\rho_1^4\frac{1}{2(2+\nu)(1+\nu)}\left(1-\frac{4}{\xi_1}\right)-\rho^4\frac{1}{2(2-\nu)(1-\nu)}\right]$$

$$-\rho_1^{-2\nu}\left[\rho_1^2\rho^2\frac{1}{1-\nu^2}\left(1-\frac{2(1-\nu)}{\xi_1}\right)+\rho_1^4\frac{1}{2(2-\nu)(1-\nu)}\right.$$

$$\left.\left.\times\left(1-\frac{2(2-\nu)}{\xi_1}\right)+\rho^4\frac{1}{2(2+\nu)(1+\nu)}\left(1+\frac{2\nu}{\xi_1}\right)\right]\right\},$$

$$D_1=A_1-B_1-C_1,\ D_2=A_2-B_2-C_2.$$

If at the initial moment of time the continuously varying functions $u(r, \tau)$, $z_1(\tau)$, $z_2(\tau)$, and $z_3(\tau)$ and their first and second derivatives are zero, we can use an operator relation between the derivative of a function and its transform [1, 3]:

$$L^{-1}\left[s^nF(s)\right]=\frac{d^nf}{d\tau^n}.$$

Then, after applying the inverse transformation to (7), and noting (15) and (16), we obtain an approximate differential equation linking the temperature of the body and the thermal effects

$$A_2\frac{d^2u}{d\tau^2}+A_1\frac{du}{d\tau}+u=B_1\frac{dz_1}{d\tau}+B_0z_1+C_1\frac{dz_2}{d\tau}+C_0z_2+\frac{a}{r_2^2}\left(D_2\frac{dz_3}{d\tau}+D_1z_3\right). \tag{27}$$

Equation (27) also holds when the initial conditions are nonzero, if sufficient time has passed since the beginning of the process and the initial thermal state has ceased to affect the temperature distribution inside the body.

The transfer functions (16) and Eq. (27) can be used to determine the typical temperatures of the body – the average temperatures $u_{S_1}(\tau)$ and $u_{S_2}(\tau)$ of the surfaces S_1 and S_2, and also the mean volume temperature $u_V(\tau)$ of the body. The form of the expressions (16) and (25) does not change, only the values of the coefficients B, C, and D change (the coefficients A_1 and A_2 are independent of ρ).

For the temperature $u_{S_1}(\tau)$ the coefficients B_{0,S_1}, B_{1,S_1}, C_{0,S_1}, C_{1,S_1}, D_{1,S_1}, and D_{2,S_1} are found from Eqs. (17)-(27) in which we have to replace ρ by ρ_1. In accordance with this, we have to put $\rho = 1$ when determining the coefficients B_{0,S_2}, B_{1,S_2}, C_{0,S_2}, C_{1,S_2}, D_{1,S_2}, and D_{2,S_2} from (20)-(26).

In the calculation of the mean volume temperature $u_V(\tau)$ the coefficients take the form

$$B_{0,V}=\frac{1}{E}\left[1-\frac{2\nu}{\xi_2}-\frac{(\nu+1)(1-\rho_1^2)}{1-\rho_1^{2\nu+2}}\right],$$

$$B_{1,V}=\frac{1}{E}\cdot\frac{r_2^2}{4a}\left\{\frac{1}{1-\nu}\left(1+\frac{2(1-\nu)}{\xi_2}\right)+\frac{1}{1-\rho_1^{2\nu+2}}\left[\frac{1}{2+\nu}(1-\rho_1^{2\nu+4})\left(1-\frac{2\nu}{\xi_2}\right)\right.\right.$$
$$\left.\left.-\frac{1+\nu}{2(1-\nu)}(1-\rho_1^4)-(1-\rho_1^2)\left(1+\frac{2}{\xi_2}\right)\right]\right\},$$

$$B_{2,V}=\frac{1}{E}\cdot\frac{r_2^4}{16a^2}\left\{\frac{1}{2(2-\nu)(1-\nu)}\left(1+\frac{2(2-\nu)}{\xi_2}\right)\right.$$
$$+\frac{1}{1-\rho_1^{2\nu+2}}\left[\frac{1}{(1-\nu)(2+\nu)}(1-\rho_1^{2\nu+4})\left(1+\frac{2(1-\nu)}{\xi_2}\right)+\frac{1}{2(2+\nu)(3+\nu)}(1-\rho_1^{2\nu+6})\left(1-\frac{2\nu}{\xi_2}\right)\right.$$
$$\left.\left.-\frac{1}{2(2+\nu)}(1-\rho_1^2)\left(1+\frac{4}{\xi_2}\right)-\frac{1}{2(1-\nu)}(1-\rho_1^4)\left(1+\frac{2}{\xi_2}\right)-\frac{1+\nu}{6(2-\nu)(1-\nu)}(1-\rho_1^6)\right]\right\},$$

$$C_{0,V}=\frac{1}{E}\left[\frac{(\nu+1)(1-\rho_1^2)}{1-\rho_1^{2\nu+2}}-\rho_1^{-2\nu}\left(1+\frac{2\nu}{\xi_1}\right)\right],$$

$$C_{1,V}=\frac{1}{E}\cdot\frac{r_2^2}{4a}\left\{\frac{1}{1-\rho_1^{2\nu+2}}\left[\rho_1^2(1-\rho_1^2)\left(1-\frac{2}{\xi_2}\right)\right.\right.$$
$$\left.\left.+\frac{\nu+1}{2(1-\nu)}(1-\rho_1^4)\right]-\rho_1^{-2\nu}\left[\rho_1^2\left(\frac{1}{1-\nu}-\frac{2}{\xi_1}\right)-\frac{1}{2+\nu}\left(1+\frac{2\nu}{\xi_1}\right)\frac{1-\rho_1^{2\nu+4}}{1-\rho_1^{2\nu+2}}\right]\right\},$$

$$C_{2,V}=\frac{1}{E}\cdot\frac{r_2^4}{16a^2}\cdot\frac{1}{(1-\rho_1^{2\nu+2})}\left[\frac{1}{2(1-\nu)}\left(1-\frac{2}{\xi_1}\right)\rho_1^2(1-\rho_1^4)\right.$$
$$\left.+\left(1-\frac{4}{\xi_1}\right)\frac{1}{2(2+\nu)}\rho_1^4(1-\rho_1^2)-\frac{\nu+1}{6(2-\nu)(1-\nu)}(1-\rho_1^6)\right]$$
$$-\left(1-\frac{2(1-\nu)}{\xi_1}\right)\frac{1}{(2+\nu)(1-\nu)}\rho_1^{2\nu+2}(1-\rho_1^{2\nu+4})$$
$$-\left(1-\frac{2(2-\nu)}{\xi_1}\right)\frac{1}{2(2-\nu)(1-\nu)}\rho_1^{2\nu+4}(1-\rho_1^{2\nu+2})-\left(1+\frac{2\nu}{\xi_1}\right)\frac{1}{2(3+\nu)(2+\nu)}(1-\rho_1^{2\nu+6}),$$

$$D_{1,V}=A_1-B_{1,V}-C_{1,V};\ D_{2,V}=A_2-B_{2,V}-C_{2,V}.$$

The transfer functions (16) and the differential equation for the influence (27) can be used to solve approximately problems in heat conduction for curvilinear walls and hollow bodies in various heat-transfer conditions when the derivation of an exact solution is difficult. We note that the coefficient A_1 approximately determines the rate of heat transfer of the body under generalized thermal influences. In stationary conditions, from (27) we obtain

$$u(\rho)=B_0z_1+C_0z_2+\frac{a}{r_2^2}D_1z_3. \tag{28}$$

It is convenient to determine the particular features of heat transfer in bodies subject to periodic thermal conditions using amplitude-phase frequency characteristics, the equations of which are obtained by replacing s in (16) by $i\omega$, where ω is the cyclic frequency of the influences.

Where necessary the accuracy of equations of the form (27) can be improved by retaining terms of higher order in the expansions of the transfer functions.

NOTATION

τ	is the time;
γ	is the density;
c	is the specific heat of the body;
λ, a	are the coefficients of thermal conductivity and thermal diffusivity of the material of the body;
α_1, α_2	are the coefficients of heat transfer for the surfaces S_1 and S_2;
$z_1(\tau)$, $z_2(\tau)$, $z_3(\tau)$	are the generalized thermal influences;
$Z_1(s)$, $Z_2(s)$, $Z_3(s)$	are the images of the influences;
T_1, T_2, V, Q_1, Q_2, W	are the Laplace transforms of the functions $t_1(\tau)$, $t_2(\tau)$, $v(\tau)$, $q_1(\tau)$, $q_2(\tau)$, and $w(\tau)$;
I_ν, K_ν	are the modified cylinder functions of arbitrary real index ν;
s	is the Laplace transform parameter;
$\rho = r/r_2$	is the relative coordinate;
$\rho_1 = r_1/r_2$.	

LITERATURE CITED

1. A. V. Lykov, The Theory of Heat Conduction [in Russian], Vysshaya Shkola, Moscow (1967).
2. N. A. Varyshev, in: Heat and Mass Transport [in Russian], Vol. 8, Nauka i Tekhnika, Minsk (1968).
3. N. A. Varyshev, The Theoretical Foundations of the Measurement of Nonstationary Temperatures [in Russian], Énergiya (1967).
4. G. Beitmen and A. Érdein, Higher Transcendental Functions [in Russian], Nauka, Moscow (1966).